Some isometry groups of Urysohn space

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Abstract

We construct various isometry groups of Urysohn space (the unique complete separable metric space that is universal and homogeneous), including abelian groups which act transitively, and free groups which are dense in the full isometry group.

1 Introduction

In a posthumously-published paper, P. S. Urysohn [6] constructed a remarkable complete separable metric space $\mathbb{U}$ that is both *homogeneous* (any isometry between finite subsets of $\mathbb{U}$ can be extended to an isometry of $\mathbb{U}$) and *universal* (every complete separable metric space can be embedded in $\mathbb{U}$). This space is unique up to isometry.

The second author [7] showed that $\mathbb{U}$ is both the generic complete metric space with distinguished countable dense subset (in the sense of Baire category) and the random such space (with respect to any of a wide class of measures).

In this paper, we investigate the isometry group $\mathrm{Aut}(\mathbb{U})$ of $\mathbb{U}$, and construct a few interesting subgroups of this group.

Our main tool is an analogous countable metric space $\mathbb{QU}$, the unique countable homogeneous metric space with rational distances. The existence and uniqueness of $\mathbb{QU}$ follow from the arguments used to establish the existence and uniqueness of $\mathbb{U}$ in [6]. Alternatively, this can be deduced from the fact that the class of finite metric spaces with rational distances has the amalgamation property, which, by Fraïssé's theorem [4], implies the result. Now $\mathbb{U}$ is the completion of $\mathbb{QU}$ (see more details in [7]). In particular, any isometry of $\mathbb{QU}$ extends uniquely to $\mathbb{U}$. Our notation suggests that $\mathbb{QU}$ is "rational Urysohn space".

Let $\mathrm{Aut}(\mathbb{QU})$ and $\mathrm{Aut}(\mathbb{U})$ be the isometry groups of $\mathbb{QU}$ and $\mathbb{U}$. We show that $\mathrm{Aut}(\mathbb{QU})$ as a subgroup of $\mathrm{Aut}(\mathbb{U})$ is dense in $\mathrm{Aut}(\mathbb{U})$ in the natural topology induced by the product topology on $\mathbb{U}^{\mathbb{U}}$). We also show that $\mathbb{QU}$ has an isometry which permutes all its points in a single cycle (indeed, it has $2^{\aleph_0}$ conjugacy classes of such isometries). The closure of the cyclic group generated by such an isometry is an abelian group which acts transitively on $\mathbb{U}$, *so that* $\mathrm{Aut}(\mathbb{U})$ *is a monothetic group and* $\mathbb{U}$ *carries an abelian group structure (in fact, many such structures).* Moreover, the free group of countable rank acts as a group of isometries of $\mathbb{QU}$ which is dense in the full isometry group (and hence is also dense in $\mathrm{Aut}(\mathbb{U})$).

The universal rational metric space $\mathbb{QU}$ is characterized by the following property: If A, B are finite metric spaces with rational distances (we say *rational metric spaces* for short) with $A \subseteq B$, then any embedding of A in $\mathbb{QU}$ can be extended to an embedding of B. It is enough to assume this in the case where $|B| = |A| + 1$, in which case it takes a more convenient form:

($*$) If A is a finite subset of $\mathbb{QU}$ and g is a function from A to the rationals satisfying

- $g(a) \geq 0$ for all $a \in A$,
- $|g(a) - g(b)| \leq d(a,b) \leq g(a) + g(b)$ for all $a, b \in A$,

then there is a point $z \in \mathbb{QU}$ such that $d(z,a) = g(a)$ for all $a \in A$.

Furthermore, $\mathbb{QU}$ is homogeneous (any isometry between finite subsets of $\mathbb{QU}$ extends to an isometry of $\mathbb{QU}$), and every countable rational metric space can be embedded isometrically in $\mathbb{QU}$.

Note that the same definition and above condition of universality of metric spaces is valid if instead of the field $\mathbb{Q}$ or $\mathbb{R}$ we consider any countable additive subgroup of $\mathbb{R}$, for example, the group of integers; in this case we have the

integer universal metric space $\mathbb{Z}\mathbb{U}$, which was considered in [1]; we will use it in Section 4; another example — the universal metric space with possible values of metric $\{0, 1, 2\}$ — is simply the universal graph Γ (see [3, 8]) with the "edge" metric on the set of its vertices.

2 $\mathrm{Aut}(\mathbb{Q}\mathbb{U})$ is dense in $\mathrm{Aut}(\mathbb{U})$

The weak topology on the group $\mathrm{Aut}(\mathbb{U})$ of isometries of $\mathbb{U}$ is that induced by the product topology on $\mathbb{U}^{\mathbb{U}}$. In particular, $g_n \to g$ if and only if, for any finite sequence $(u_1, \ldots, u_m)$ of points and any $\epsilon > 0$, there exists n_0 such that $d(g_n(u_i), g(u_i)) < \epsilon$ for $1 \leq i \leq m$ and $n \geq n_0$.

Theorem 1 *The group* $\mathrm{Aut}(\mathbb{Q}\mathbb{U})$ *is a dense subgroup of* $\mathrm{Aut}(\mathbb{U})$ *in weak topology.*

It suffices to show the following property of $\mathbb{Q}\mathbb{U}$:

Proposition 2 *Given* $\epsilon > 0$ *and* $v_1, \ldots, v_n, v'_1, \ldots, v'_{n-1}, v''_n \in \mathbb{Q}\mathbb{U}$ *such that* $(v_1, \ldots, v_{n-1})$ *and* $(v'_1, \ldots, v'_{n-1})$ *are isometric and*

$$|d(v'_i, v''_n) - d(v_i, v_n)| < \epsilon,$$

there exists $v'_n \in \mathbb{Q}\mathbb{U}$ *such that* $(v_1, \ldots, v_n)$ *and* $(v'_1, \ldots, v'_n)$ *are isometric and* $d(v'_n, v''_n) < \epsilon$.

Proof Assuming this for a moment, we complete the proof of the density as follows. We are given an isometry g of $\mathbb{U}$ and points $u_1, \ldots, u_m \in \mathbb{U}$. Choose points $v_1, \ldots, v_m \in \mathbb{Q}\mathbb{U}$ with $d(v_i, u_i) < \epsilon/4m$. Now using the above proposition, we inductively choose points $v'_1, \ldots, v'_m$ so that $(v_1, \ldots, v_m)$ and $(v'_1, \ldots, v'_m)$ are isometric and $d(v'_i, g(u_i)) < i\epsilon/m$. For suppose that $v'_1, \ldots, v'_{n-1}$ have been chosen. Choose any point $v''_n \in \mathbb{Q}\mathbb{U}$ with $d(g(u_n), v''_n) < \epsilon/4m$. Then

$$d(u_i, u_n) - \epsilon/2m < d(v_i, v_n) < d(u_i, u_n) + \epsilon/2m,$$

and

$$d(g(u_i), g(u_n)) - (4i + 1)\epsilon/4m < d(v'_i, v''_n) < d(g(u_i), g(u_n)) + (4i + 1)\epsilon/4m,$$

so

$$|d(v'_i, v''_n) - d(v_i, v_n)| < (4i + 3)\epsilon/4m \leq (4n - 1)\epsilon/4m.$$

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So we may apply the proposition to choose v'_n with $d(v'_n, v''_n) < (4n-1)\epsilon/4m$. Then $d(v'_n, g(u_n)) < n\epsilon/m$, and we have finished the inductive step. At the conclusion, we have $d(v'_n, g(u_n)) < n\epsilon/m \leq \epsilon$ for $1 \leq n \leq m$.

Now we find an isometry of $\mathbb{QU}$ mapping v_i to v'_i for $1 \leq i \leq m$ (by the homogeneity of $\mathbb{QU}$), and the proof is complete. □

Proof of Proposition 1. We have to extend the set $\{v'_1, \ldots, v'_{n-1}, v''_n\}$ by adding a point v'_n with prescribed distances to $v'_1, \ldots, v'_{n-1}$ and distance less than ϵ to v''_n. So it is enough to show that these requirements do not conflict, that is, that

$$|d(v'_n, x) - d(v'_n, y)| \leq d(x, y) \leq d(v_n\, x) + d(v_n\, y)$$

for $x, y \in \{v'_1, \ldots, v'_{n-1}, v''_n\}$. There are no conflicts if $x, y \neq v''_n$: this follows from the fact that the points $v_1, \ldots, v_n$ exist having the required distances. So we may assume that $x = v'_i$ and $y = v''_n$, in which case the consistency follows from the hypothesis. □

3 $\mathrm{BAut}(\mathbb{U})$ **is dense in** $\mathrm{Aut}(\mathbb{U})$

For a metric space M, we define $\mathrm{BAut}(M)$ to be the group of all *bounded* isometries of M (those satisfying $d(x, g(x)) \leq k$ for all $x \in M$, where k is a constant). Clearly it is a normal subgroup of $\mathrm{Aut}(M)$, though in general it may be trivial, or it may be the whole of $\mathrm{Aut}(M)$.

We show that $\mathrm{BAut}(\mathbb{QU})$ is a dense subgroup of $\mathrm{Aut}(\mathbb{QU})$: in other words, any isometry between finite subsets of $\mathbb{QU}$ can be extended to a bounded isometry of $\mathbb{QU}$. This is immediate from the following lemma.

Lemma 3 *Let f be an isometry between finite subsets A and B of $\mathbb{QU}$, satisfying $d(a, f(a)) \leq k$ for all $a \in A$. Then f can be extended to an isometry g of $\mathbb{QU}$ satisfying $d(x, g(x)) \leq k$ for all $x \in \mathbb{QU}$.*

Proof Suppose that $f : a_i \mapsto b_i$ for $i = 1, \ldots, n$, with $d(a_i, b_i) \leq k$. It is enough to show that, for any point $u \in \mathbb{QU}$, there exists $v \in \mathbb{QU}$ such that $d(b_i, v) = d(a_i, u)$ for all i and $d(u, v) \leq k$. For then we can extend f to any further point; the same result in reverse shows that we can extend f^{-1}, and then we can construct g by a back-and-forth argument.

The pont v must satisfy $d(b_i, v) = d(a_i, u)$ and $d(u, v) \leq k$. We must show that these requirements are consistent; then the existence of v follows from the extension property of $\mathbb{QU}$. Clearly the consistency conditions for the values $d(b_i, v)$ are satisfied. So the only possible conflict can arise from the inequality

$$|d(v, u) - d(v, b_i)| \leq d(u, b_i) \leq d(v, u) + d(v, b_i).$$

We wish to impose an upper bound on $d(v, u)$, so a conflict could arise only if a lower bound arising from the displayed equation were greater than k, that is, $|d(v, b_i) - d(u, b_i)| > k$, or equivalently, $|d(u, a_i) - d(u, b_i)| > k$. But this is not the case, since

$$|d(u, a_i) - d(u, b_i)| \leq d(a_i, b_i) \leq k.$$

$\square$

4   A cyclic isometry of universal spaces $\mathbb{U}$.

Universal Urysohn metric space has very important and rather surprising property which we formulate in the following theorem:

Theorem 4 *There is an isometry $g \in ISO(\mathbb{U})$ of the universal Urysohn space $\mathbb{U}$ such that the $\langle g \rangle$-orbit of some point x (i.e., the set $\{g^n x; n \in \mathbb{Z}\}$) is dense in $\mathbb{U}$; in this case the $\langle g \rangle$-orbit of each point is dense in $\mathbb{U}$.*

The second claim follows from the first one directly. One of the important corollaries of this theorem is

Proposition 5 *There are transitive abelian groups of isometries of $\mathbb{U}$ of infinite exponent.*

Proof   Let $\overline{G}$ be the closure in $\mathrm{Aut}(\mathbb{U})$ of the cyclic group $\langle g \rangle$. Since the orbits of g are dense, it is clear that $\overline{G}$ is transitive. Moreover, as the closure of an abelian group, it is itself abelian. For, if $h, k \in \overline{G}$, say $h_i \to h$ and $k_i \to k$; then $h_i k_i = k_i h_i \to hk = kh$. $\square$

What is the structure of the group $\overline{G}$ — the closure of the group $\mathbb{Z} = \{g^n, n \in \mathbb{Z}\}$? Since there are many choices for such g, we must expect that their closures will not all be alike. In particular, there should be some choices of g such that $\overline{G} = G(g)$ is torsion-free, and others for which it is not. What is very important is the fact that the Urysohn space can be equipped with the structure of an abelian group; this must help to find some appropriate model of this space; we will discuss this question elsewhere.

The proof of Theorem 4 follows easily from the analogous fact for the rational case:

Theorem 6 *There is an isometry g of $\mathbb{Q}\mathbb{U}$ such that $\langle g \rangle$ is transitive on $\mathbb{Q}\mathbb{U}$.*

Indeed, since the completion of the space $\mathbb{Q}\mathbb{U}$ is the universal metric space $\mathbb{U}$, a transitive isometry of $\mathbb{Q}\mathbb{U}$ extends to an isometry of $\mathbb{U}$ with dense orbit.

We can also put the same question about the universal homogeneous *integer* metric space $\mathbb{Z}\mathbb{U}$; the answer is as follows:

Theorem 7 (see [1]) *There exists a transitive isometry of the universal integer metric space $\mathbb{Z}\mathbb{U}$.*

As we will see, Theorem 6 on the rational case is a corollary of Theorem 7 on the existence of universal integer metric space. The latter fact was discovered in [1]; to make this paper self-contained, and for its own interest, below we give the proof, which is similar to the considerations from [1].

We must define a metric with integer, rational, or real values on the set of integers $\mathbb{Z}$ that is shift-invariant. We will call it a cyclic metric. Such a metric is completely determined by the function $f(i) = d(i, 0)$ on the non-negative integers; for $d(i, j) = f(|j - i|)$. The function should satisfy the constraints

(a) $f(i) \geq 0$, with equality if and only if $i = 0$.

(b) $|f(i) - f(j)| \leq f(i + j) \leq f(i) + f(j)$ for all i, j.

We call a function satisfying (a) and (b) a *Toeplitz distance function*, and denote the set of such functions by $\mathbb{Q}T$ (resp. $\mathbb{Z}T$) if the values of f are rational (resp. integer). If i, j in (a), (b) run over $1, \ldots, n$ only, then the set of such functions will be denoted by $\mathbb{Q}T_n$ ($\mathbb{Z}T_n$).

Now the cyclic metric space given by such a function is isometric to the universal space $\mathbb{Q}\mathbb{U}$ if and only if f has the following property:

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(c) given any function h from $\{1,\ldots,k\}$ to the positive rationals satisfying

$$|h(i) - h(j)| \le f(|i - j|) \le h(i) + h(j)$$

for $i,j \in \{1,\ldots,k\}$, there exists a positive integer N such that $h(i) = f(N - i)$ for all $i \in \{1,\ldots,k\}$.

We say that a Toeplitz distance function is *universal* if it satisfies (c). The same criterion of universality is true for metrics with integer values. It is convenient in the proof to use integer metrics instead of rational ones.

Given $f \in \mathbb{Q}T_n$ (or $\mathbb{Z}T_n$), we say that an m-tuple $(h(1),\ldots,h(m))$ is *f-admissible* if

$$|h(i) - h(i + k)| \le f(k) \le h(i) + h(i + k)$$

for $1 \le i < i + k \le m$ and $k \le n$. This notion of admissibility agrees with the admissibility of vectors with respect to the distance matrix $\{d(i,j)\}_{i,j=1}^n$ in the sense of [7]. Note that if h is f-admissible, then it is admissible with respect to the restriction of f to $\{1,\ldots,n'\}$ for any $n' \le n$.

In order to prove the existence of a cyclic isometry of the universal rational or integer metric space, it is enough to prove the following theorem.

Theorem 8 *There exists a function $F = (f(0) = 0, f(1),\ldots) \in \mathbb{Q}T$ with the following property: for each n and for each F^n-admissible vector $\{h(i)\}, i = 1,\ldots,n$ (where $F^n = (f(0),\ldots,f(n)) \in \mathbb{Q}T_n$ is the initial fragment of F of length n), there exists N such that $f(N + i) = h(i), i = 1,\ldots,n$.*

Indeed, such a function F satisfies condition (c) above, i.e., F is a universal Toeplitz function; and it defines a rational metric on $\mathbb{Z}$ with required property.

In order to prove Theorem 8, we will construct such a function F by induction: we will define it by a recursive procedure, each step being founded on the following theorem:

Theorem 9 *For any given finite function $F_n = (f(i))_{i=0}^n \in \mathbb{Q}T_n$ (or $\mathbb{Z}T_n$) and any vector $H_n = (h(i))_{i=1}^n$ that is admissible with respect to F_n, there exist a positive integer m and a vector $G = (g(1),\ldots,g(m))$ such that the function $\bar{F} = (f(0), f(1),\ldots, f(n), g(1),\ldots, g(m), h(1),\ldots, h(n))$ (the prolongation of F_n) belongs to $\mathbb{Q}T_{2n+m}$ (or $\mathbb{Z}T_{2n+m}$).*

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Using this fact, the required universal function $F \in \mathbb{Q}T$ (or $F \in \mathbb{Z}T$) can be easily obtained as follows. Enumerate all finite integer vectors $V = \{v_1, v_2, \ldots\}$; choose an initial function $F_0 \in \mathbb{Q}T_{n_0}$ and then find the first vector from the sequence V that is admissible for F_0 and apply the theorem giving the prolongation of F_0; if we already have some finite F_n, we choose the next admissible vector from V and apply the prolongation procedure. As a result, we obtain an infinite universal function F.

It suffices to prove Theorem 9 above for an integer function $F_n \in \mathbb{Z}T_n$, because before applying the prolongation procedure we can multiply the given rational vector $F \in \mathbb{Q}T_n$ by a suitable integer and obtain $F_n \in \mathbb{Z}T_n$, and then, after the prolongation procedure, divide the prolongation by the same integer. So we will give a proof of Theorem 8 for the case $F_n \in \mathbb{Z}T_n$.

The construction of prolongation is based on several lemmas, the first one being the simple *amalgamation lemma*, which is an analog of the lemma from [8] and [1].

Lemma 10 *For each $F_n \in \mathbb{Q}T_n$, let*

$$M(F_n) = \max_{1 \le k \le n} |f(k) - f(n-k+1)|, \qquad m(F_n) = \min_{1 \le k \le n} (f(k) + f(n-k+1)).$$

Then $M(F_n) \le m(F_n)$.

Proof We have $f(k) + f(n-k) + f(n-j) = d(0,k) + d(0,n-k) + d(n,j) = d(0,k) + d(n,k) + d(n,j) \ge d(0,k) + d(k,j) \ge d(0,j) = f(j)$, so $f(k) + f(n-k) \ge f(j) - f(n-j)$ for all $j, k = 1, \ldots, n$. Consequently,

$$m(F_n) \equiv \min_k [f(k) + f(n-k)] \ge \max_j |f(j) - f(n-j)| \equiv M(F_n).$$

$\square$

The next lemma shows how to extend a given Toeplitz function and a given admissible vector with one coordinate each, namely, how to join a new coordinate to the function and a new coordinate to the beginning of the admissible vector. It will be the base of induction.

Lemma 11 *Suppose we have a function $F \equiv F_n = (f(i))_{i=1}^n \in \mathbb{Z}T_n$[1] and an F_n-admissible integer vector $H \equiv H_n = (h(i))_{i=1}^n$. Set*

$$d = \max_{i=1,\ldots,n} \{f_i, h_i\}.$$

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[1]We omit the coordinate $f(0)$, which is identically equal to 0.

Then there exist two integers $g(1), g(N)$ such that

(i) *the function $F_{n+1} = (f(1), \ldots, f(n), g(1))$ lies in $\mathbb{Z}T_{n+1}$;*

(ii) *the vector $H_{n+1} = (g(N), h(1), \ldots, h(n))$ is F_{n+1}-admissible;*

(iii) $2 \leq g(1) \leq d - 1$, $2 \leq g(N) \leq d - 1$.

Proof The numbers $g(1)$ and $g(N)$ are solutions of the following system of linear inequalities:

$$\max_{i=1,\ldots,n} |f(i) - f(n-i)| \leq g(1) \leq \min_{i=1,\ldots,n} (f(i) + f(n-i)) \tag{1}$$

$$\max_{i=1,\ldots,n} |f(i) - h(i)| \leq g(N) \leq \min_{i=1,\ldots,n} (f(i) + h(i)), \tag{2}$$

$$|g_1 - g_N| \leq h_n \leq (g(1) + g(N)), \tag{3}$$

$$2 \leq g(1) \leq d - 1, \quad 2 \leq g(N) \leq d - 1. \tag{4}$$

Inequality (1) expresses the fact that the extended function $F' \equiv (f(1), \ldots, f(n), g(1))$ belongs to $\mathbb{Z}T_n$; inequality (2) means that the extended vector $(g(N), h(1), \ldots, h(n))$ is F'-admissible; inequality (3), together with inequality (1) means the F'-admissibility of the coordinate $h(n)$. The compatibility of inequalities (1)–(3) is a corollary of the amalgamation lemma and the F-admissibility of h, which is easy to check; condition (4) follows from the definition of d and easy calculations. So the required properties (i), (ii), (iii) for this choice of $g(1)$ and $g(N)$ are fulfilled. $\square$

The construction of the vector $(g_1, \ldots, g_m)$ is as follows. First of all, we put $m = nd$ and define a prolongation of F as a vector divided into d blocks, each of length n:

$(F, G_1, \ldots, G_d, H)$ where $G_i = (g((i-1)n + 1), \ldots, g(in)); \quad i = 1, \ldots, d.$

It is convenient to denote $F = G_0$ and $H = G_{d+1}$; the coordinates $f(i)$, $h(i)$ belong to the integer interval $[1, d]$. We will define vectors G_i, $i = 1, \ldots, d$ successively in the following order ("from both sides"): G_1, G_d, then G_2, G_{d-1}, and so on up to $G_{\frac{d-3}{2}}, G_{\frac{d+1}{2}}$, and, finally, $G_{\frac{d-1}{2}}$. Now let us define G_1 and G_2.

We apply Lemma 11 and obtain $g(1)$ and $g(N) \equiv g(nd)$. The condition of the lemma is fulfilled for the extended vector, and we can repeat the same procedure with the extended vectors $F' = (f(1), \ldots, f(n), g(1))$ and $H' = (g(nd), h(1), \ldots, h(n))$ and join numbers $g(2)$ and $g(nd-1)$, obtaining vectors $(f(1), \ldots, f(n), g(1), g(2))$ and $= (g(nd-1), g(nd), h(1), \ldots, h(n))$; then we join $g(3)$ and $g(nd-2)$, etc., up to $g(n)$ and $g((n-1)d+1)$. All

these integers belong to the interval $[2, d-1]$. As a result, we obtain the first part of the construction, namely, the vectors G_1, G_d. We continue in the same way and apply Lemma 10 to the vectors G_1, G_2 (instead of F, H) with only one change: the integer coordinates $g(n+1), g(n+2), \ldots, g(2n)$ and $g((n-1)d), \ldots, g((n-2)d+1)$ belong to the interval $[3, d-2]$ (instead of $[2, d-1]$ in the case of G_1, G_d). Thus the interval shrinks. By induction, we obtain all blocks up to the last block $G_{\frac{d-1}{2}}$, whose all coordinates are equal: $g(n(\frac{d-1}{2}+i)) = \frac{d-1}{2}$ (because the interval is reduced to the single point $\frac{d-1}{2}$). So we join the beginning and the end and obtain the required vector $G_0 = F, G_1, \ldots, G_d, H$. $\square$

This gives the proof of Theorems 8, 9 and the existence of a transitive isometry on $\mathbb{Z}\mathbb{U}, \mathbb{Q}\mathbb{U}$ and an isometry on $\mathbb{U}$ with dense orbit.

The proof of the theorem gives also further information:

Corollary 12 *The group* $\operatorname{Aut}(\mathbb{Q}\mathbb{U})$ *contains* $2^{\aleph_0}$ *conjugacy classes of isometries which permute the points in a single cycle. Moreover, representatives of these classes remain non-conjugate in* $\operatorname{Aut}(\mathbb{U})$.

Proof It is clear that, if cyclic isometries g and h are conjugate, then the functions f_g and f_h describing them as in the above proof are equal. For, if $h = k^{-1}gk$, then

$$f_h(n) = d(x, h^n(x)) = d(x, k^{-1}g^n k(x)) = d(k(x), g^n k(x)) = f_g(n).$$

But the set of functions describing cyclic isometries of $\mathbb{Q}\mathbb{U}$ is residual, hence of cardinality $2^{\aleph_0}$. $\square$

The cyclic isometries constructed in this section have the property that $d(x, g(x))$ is constant for $x \in \mathbb{Q}\mathbb{U}$, and hence this holds for all $x \in \mathbb{U}$. In particular, these isometries are bounded.

5   An abelian group of exponent 2

To extend this argument to produce other groups acting regularly on $\mathbb{Q}\mathbb{U}$, it is necessary to change the definition of a Toeplitz function so that the metric is defined by translation in the given group. We give here one simple example.

Proposition 13 *The countable abelian group of exponent* 2 *acts regularly as an isometry group of* $\mathbb{Q}\mathbb{U}$.

Proof This group G has a chain of subgroups $H_0 \leq H_1 \leq H_2 \leq \cdots$ whose union is G, with $|H_i| = 2^i$. We show that, given any H_i-invariant rational metric on H_i and any $h \in H_{i+1} \setminus H_i$, we can prescribe the distances from h to H_i arbitrarily (subject to the consistency condition) and extend the result to an H_{i+1}-invariant metric on H_{i+1}. The extension of the metric is done by translation in H_{i+1}: note that $H_{i+1} \setminus H_i$ is isometric to H_i, since $d(h + h', h + h'') = d(h', h'')$ for $h', h'' \in H_i$. Now the resulting function is a metric. All that has to be verified is the triangle inequality. Now triangles with all vertices in H_i, or all vertices in $H_{i+1} \setminus H_i$, clearly satisfy the triangle inequality. Any other triangle can be translated to a triangle containing h and two points of H_i, for which the triangle inequality is equivalent to the consistency condition for extending the metric to $H_i \cup \{h\}$. $\qquad \square$

Note that almost all G-invariant metrics (in the sense of Baire category) are isometric to $\mathbb{Q}\mathbb{U}$.

We can construct the analogous transitive actions for other abelian groups on $\mathbb{U}$.

Proposition 14 *There are transitive abelian groups of isometries of* $\mathbb{U}$ *of the groups of exponent* 2.

Proof Let G be one of the abelian groups previously constructed, and $\overline{G}$ its closure in Aut($\mathbb{U}$). Since the orbits of G are dense, it is clear that $\overline{G}$ is transitive. Moreover, as the closure of an abelian group, it is itself abelian. For, if $h, k \in \overline{G}$, say $h_i \to h$ and $k_i \to k$; then $h_i k_i = k_i h_i \to hk = kh$. Similarly, if G has exponent 2, then so does $\overline{G}$. $\qquad \square$

6   Other regular group actions

There is a one-way relationship between transitive group actions on Q (or, more generally, group actions on $\mathbb{U}$ with a dense orbit) and transitive actions on the Universal (random) graph R, as given in the following result.

Proposition 15 *Let G be a group acting on Urysohn space $\mathbb{U}$ with a countable dense orbit X. Then there exists a natural structure of the universal graph R on X and group G preserves this structure.*

Proof Partition the positive real numbers into two subsets E and N such that, for any $R, \epsilon > 0$, there are consecutive intervals of length at most ϵ to the right of R with one contained in E and the other in N. (For example, take a divergent series (a_n) whose terms tend to zero, and put half-open intervals of length a_n alternately in E and N.)

We define a graph on X by letting $\{x, y\}$ be an edge if $d(x, y) \in E$, and a non-edge if $d(x, y) \in N$. Clearly this graph is G-invariant; we must show that it is isomorphic to the random graph R.

Let U and V be finite disjoint sets of points of X, and let the diameter of $U \cup V$ be d and the minimum distance between two of its points be m. Choose $R > d/2$ and $\epsilon < m/2$, and find consecutive intervals I_E and I_N as above. Let $U \cap V = \{w_1, \ldots, w_n\}$. Choosing arbitrary values $g(i) \in I_E \cap I_n$, the consistency condition

$$|g(i) - g(j)| \le d(w_i, w_j) \le g(i) + g(j)$$

is always satisfied. So choose the values such that $g(i)$ is in the interior of I_E if $w_i \in U$, and in the interior of I_n if $w_i \in V$. Let z be a point of U with $d(z, w_i) = g(i)$. Since X is dense, we can find $x \in X$ such that $d(x, z)$ is arbitrarily small; in particular, so that $d(x, w_i)$ is in I_E (resp. I_N) if and only if $d(z, w_i)$ is. Thus x is joined to all vertices in U and to none in V. This condition characterizes R as a countable graph. $\square$

The converse is not true. A special case of the result of Cameron and Johnson [2] shows that a sufficient condition for a group G to act regularly on R is that any element has only finitely many square roots. In a group with odd exponent, each element has a unique square root. So any such group acts regularly on R. But we have the following:

Proposition 16 *The countable abelian group of exponent 3 cannot act on $\mathbb{U}$ with a dense orbit, and in particular cannot act transitively on $\mathbb{QU}$.*

Proof Suppose that we have such an action of this group A. Since the stabiliser of a point in the dense orbit is trivial, we can identify the points of the orbit with elements of A (which we write additively).

Choose $x \ne 0$ and let $d(0, x) = \alpha$. Then $\{0, x, -x\}$ is an equilateral triangle with side α. Since $\mathbb{U}$ is universal and A is dense, there is an element y such that $d(x, y), d(-x, y) \approx \frac{1}{2}\alpha$ and $d(0, y) \approx \frac{3}{2}\alpha$. (The approximation is to within a given ϵ chosen smaller than $\frac{1}{6}$. Then the three points $0, y, x - y$ form a triangle with sides approximately $\frac{3}{2}\alpha$, $\frac{1}{2}\alpha$, $\frac{1}{2}\alpha$, contradicting the triangle inequality. $\square$

7 Unbounded isometries of $\mathbb{U}$

The subgroup BAut($\mathbb{U}$) is not the whole isometry group, because unbounded isometries exist. The simplest way to see this is to mention that Euclidean space R^n can be imbedded to $\mathbb{U}$ in such a way that the group of motions $Iso(R^n$ has monomorphic imbedding to $Iso(\mathbb{U})$. But we will give a direct construction of such isometry. (We are grateful to J.Nešetřil for the following argument.)

Proposition 17 *There exist unbounded isometries of $\mathbb{QU}$ (and hence of $\mathbb{U}$).*

The proof depends on a lemma.

Lemma 18 *Let A be a finite subset of $\mathbb{QU}$ and let g be a function on A satisfying the consistency conditions $(*)$. Then the diameter of the set*

$$S_A = \{z \in \mathbb{QU} : d(z, a) = g(a) \text{ for all } a \in A\}$$

is twice the minimum value of g.

Proof Let z_1 and z_2 be two points from S_A. Consider the problem of adding z_2 to the set $A \cup \{z_1\}$. The consistency conditions for z_2 are precisely those for z_1 together with the conditions

$$|d(z_2, z_1) - d(z_2, a)| \leq d(z_1, a) \leq d(z_2, z_1) + d(z_2, a)$$

for all $a \in A$. Since $d(z_1, a) = d(z_2, a) = g(a)$, the only non-trivial restriction is $d(z_1, z_2) \leq 2d(z_1, a) = 2g(a)$, which must hold for all $a \in A$. $\square$

Proof of Proposition 17 We construct an isometry f of $\mathbb{QU}$ by the standard back-and-forth method, starting with any enumeration of $\mathbb{QU}$. At odd-numbered stages we choose the first point not in the range of f and select a suitable pre-image. At stages divisible by 4 we choose the first point not in the domain of f and select a suitable image. This guarantees that the isometry we construct is a bijection from $\mathbb{QU}$ to itself.

At stage $4n + 2$, let $\mathbb{U}$ be the domain of f. Choose an unused point z whose least distance from $\mathbb{U}$ is n. Now the diameter of the set of possible images of z is $2n$; so we can choose a possible image $f(z)$ whose distance from z is at least n. Then the constructed isometry is not bounded. $\square$

We can improve this argument to construct an isometry g such that all powers of g except the identity are unbounded. In fact, even more is true:

Lemma 19 *There are two isometries a, b of $\mathbb{QU}$ which generate a free group, all of whose non-identity elements are unbounded isometries.*

Proof We begin by enumerating $\mathbb{QU} = (x_0, x_1, \ldots)$. We follow the argument we used to show that unbounded isometries exist. We construct a and b simultaneously, using the even-numbered stages for a back-and-forth argument to ensure that both are bijections, and the odd-numbered stages to ensure that any word in a and b is unbounded. The first requirement is done as we have seen before.

Enumerate the words $w(a, b)$ in a and b and their inverses. (It suffices to deal with the cyclically reduced words, since all others are conjugates of these.) We show first how to ensure that $w(a, b) \neq 1$. At a given stage, suppose we are considering a word $w(a, b)$. Choose a point x_i such that neither a nor b, nor their inverses, has been defined on x_j for $j \geq i$. Suppose that w ends with the letter a. Since there are infinitely many choices for the image of x_i under a, we may choose an image x_j with $j > i$. Now define the action of the second-last letter of the word on x_j so that the image is x_k with $k > j$. Continuing in this way, we end up with a situation where $w(a, b)x_i = x_m$ with $m > i$. So $w(a, b) \neq 1$.

To ensure that $w(a, b)$ is unbounded, we must do more. Enumerate the words so that each occurs infinitely often in the list. Now, the kth time we revisit the word w, we can ensure (as in our construction of an unbounded isometry) that $d(x_i, w(a, b)x_i) \geq k$. Thus $w(a, b)$ is unbounded. $\square$

8 A dense free subgroup of $\mathrm{Aut}(\mathbb{U})$

We can now use a trick due to Tits [5] to show that there is a dense subgroup of $\mathrm{Aut}(\mathbb{U})$ which is a free group of countable rank.

Theorem 20 *There is a subgroup F of $\mathrm{Aut}(\mathbb{QU})$ which acts faithfully and homogeneously on $\mathbb{QU}$ and is isomorphic to the free group of countable rank.*

Proof Since the free group F_2 contains a subgroup isomorphic to F_ω, choose a group H with free generators h_i for $i \in \mathbb{N}$, such that $H \cap \mathrm{BAut}(\mathbb{QU}) = 1$. Enumerate the pairs of isometric n-tuples of elements of $\mathbb{QU}$, for all n, as

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(α_0, β_0), (α_1, β_1), ... Now, for each i, Lemma 3 shows that we can choose $n_i \in \mathrm{BAut}(\mathbb{Q}\mathbb{U})$ such that $n_i h_i(\alpha_i) = \beta_i$. Let F be the group generated by the elements $n_1 h_1, n_2 h_2, \ldots$. Clearly F acts homogeneously on $\mathbb{Q}\mathbb{U}$. We claim that F is free with the given generators. Suppose that $w(n_i h_i) = 1$ for some word w. Since $\mathrm{BAut}(\mathbb{Q}\mathbb{U})$ is a normal subgroup, we have $nw(h_i) = 1$ for some $n \in \mathrm{BAut}(\mathbb{Q}\mathbb{U})$. Since n is bounded and $w(h_i)$ unbounded, this is impossible. In fact this argument shows that all the non-identity elements of F are unbounded isometries. $\square$

References

[1] P. J. Cameron, Homogeneous Cayley objects, *Europ. J. Combinatorics* **21** (2000).

[2] P. J. Cameron and K. W. Johnson, An essay on countable B-groups, *Math. Proc. Cambridge Philos. Soc.* **102** (1987), 223–232.

[3] P. J. Cameron, The random graph, *The mathematics of Paul Erdos II*, R. L. Graham and J. Nesetril (eds.). Springer-Verlag, Berlin, 1997, pp. 333–351.

[4] R. Fraïssé, Sur certains relations qui généralisent l'ordre des nombres rationnels, *C. R. Acad. Sci. Paris* **237** (1953), 540–542.

[5] J. Tits, *Buildings of Spherical Type and Finite BN-Pairs*, Lecture Notes in Math. **382**, Springer–Verlag, Berlin, 1974.

[6] P. S. Urysohn, Sur un espace metrique universel, *Bull. Sci. Math.* **51** (1927), 1–38.

[7] A. M. Vershik, A random metric space is a Uryson space (Russian), *Dokl. Akad. Nauk* **387** (2002), 733–736.

[8] A. M. Vershik, The universal and random metric spaces, *Russian Math. Surv.* **356** (2004), No. 2, 65–104.